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A Copula-Based BRDF Model

Aydın Öztürk[1], Murat Kurt[2] and Ahmet Bilgili[2]

[1]Department of Computer Engineering, Yasar University, Turkey
aydin.ozturk@yasar.edu.tr
[2]International Computer Institute, Ege University, Turkey
murat.kurt@ege.edu.tr, ahmetbilgili@gmail.com

Abstract

In this paper, we introduce a novel approach for modeling surface reflection. We focus on using a family of probability distributions called Archimedean copulas as BRDF models. The Archimedean representation has an attractive property in that the multivariate distributions are characterized by their marginal distributions through a single univariate function only. It is shown that the proposed model meets the reciprocity property of reflection. Based on measured BRDF data, we demonstrate that the proposed approach provides a good approximation to BRDF. Empirical comparisons are made with some classically used BRDF models.

Keywords: BRDF representation, reflection models, rendering, copula distributions

Categories and Subject Descriptors (according to ACM CCS): I.3.7 [Computer Graphics]: Three-Dimensional Graphics and Realism

1. Introduction

The Bidirectional Reflectance Distribution Function (BRDF) describes the surface reflectance of a material [NRH*77]. It is defined in terms of incoming and outgoing radiances by the following expression:

$$L_o(\boldsymbol{\omega_o}) = \int_\Omega L_i(\boldsymbol{\omega_i})\rho(\boldsymbol{\omega_i}, \boldsymbol{\omega_o})(\boldsymbol{\omega_i} \cdot \mathbf{n})d\boldsymbol{\omega_i}, \qquad (1)$$

where L_o and L_i are the outgoing and incoming radiances; $\boldsymbol{\omega_o}$ and $\boldsymbol{\omega_i}$ are the corresponding direction vectors, respectively; $\rho(\boldsymbol{\omega_i}, \boldsymbol{\omega_o})$ is the BRDF; $\mathbf{n}$ is the normal vector; and Ω represents the hemisphere of incoming light directions.

In the aforementioned equation, it is difficult to formulate an exact mathematical expression for the BRDF $\rho(\boldsymbol{\omega_i}, \boldsymbol{\omega_o})$ of a given realistic material. However, a wide range of BRDF models with varying degrees of complexity have been proposed to approximate surface reflectance. Some of these models are designed to represent the reciprocity and energy-conserving properties of reflection. Some other models are phenomenological, that is, their certain features of reflection are described by choosing appropriate functions. Another approach for describing surface reflectance is to measure the BRDF of real materials and fit a known model to this data.

The energy conserving property of an ideal BRDF dictates that for every outgoing light direction

$$\int_\Omega \rho(\boldsymbol{\omega_i}, \boldsymbol{\omega_o})(\boldsymbol{\omega_i} \cdot \mathbf{n})d\boldsymbol{\omega_i} \leq 1. \qquad (2)$$

For a given direction $\boldsymbol{\omega_o}$ the function

$$f(\boldsymbol{\omega_i}, \boldsymbol{\omega_o}) = \rho(\boldsymbol{\omega_i}, \boldsymbol{\omega_o})(\boldsymbol{\omega_i} \cdot \mathbf{n}), \qquad (3)$$

can be viewed as a multivariate probability density function (pdf) if the BRDF satisfies the equality in the aforementioned equation. Edwards et al. [EBJ*06] presented a method based on representing halfway vector distributions in two-dimensional (2D) domains to enforce energy conservation in a BRDF.

Since $0 \leq (\boldsymbol{\omega_i} \cdot \mathbf{n}) \leq 1$, the inequality in Equation (2) still holds when the BRDF function $\rho(\boldsymbol{\omega_i}, \boldsymbol{\omega_o})$ can be treated as a pdf. In this paper, we use a scaling factor for the measured

BRDF which allows us to treat the corresponding model in the context of multivariate probability distributions. Thus an appropriate probability distribution can be employed as a BRDF model for which reciprocity property of reflection is satisfied. Based on this consideration, the function $\rho(\boldsymbol{\omega_i}, \boldsymbol{\omega_o})$ can be viewed as a four-dimensional (4D) multivariate pdf when the direction vectors are defined in terms of the standard spherical coordinates that is $\boldsymbol{\omega_o} = (\theta_o, \phi_o)$ and $\boldsymbol{\omega_i} = (\theta_i, \phi_i)$ where θ and ϕ represent the elevation and the azimuth angles, respectively. In this case, the elevation and the azimuth angles for incident and reflection directions are considered to be random variables and the corresponding BRDF as the joint pdf of these random variables. Clearly any representation, such as the one based on the halfway vector instead of the standard spherical coordinates, can also be used to define the underlying multivariate distribution [Rus98, APS00]. If the joint distribution of the random variables to be considered were known, then the problem of describing the BRDF of a given material would consist of estimating the parameters of the corresponding distribution.

The best known and most important multivariate distribution is the multivariate normal distribution (MVN). An attractive property of the MVN distribution is that it can be fully characterized by the marginal distributions and the correlation coefficients between any two random variables. Some other multivariate distributions also have been derived from the MVN distribution. The normal distribution has been used in the modeling of reflectance [CT81, War92, APS00]. A major disadvantage of using MVN for modeling BRDF is that each marginal distribution has to be a normal distribution.

Various families of multivariate distributions have been developed for certain applications where the MVN distribution does not provide a satisfactory approximation. Application of these distributions is limited by the lack of generality that a different family is required for each set of marginal distributions.

An interesting approach to constructing multivariate distributions with a number of attractive properties is based on the so-called *copula* function. The copula approach is a modeling strategy where a joint distribution function is defined through a function, namely a copula, by specifying marginal distributions. In other words, it is a function that links the marginal distributions to their complete multivariate distribution. The copula representation not only maintains the dependence structure of the random variables, thereby capturing all of the joint behavior, but also provides great flexibility through the margins.

A main result based on Sklar's theorem [Nel06] is that if the marginal distributions are continuous then there exists a unique copula representation for a p-dimensional multivariate distribution. In this sense, the copulas provide a unifying approach to modeling surface reflection.

Although the copula approach is a relatively new method, there has been a rapidly growing interest in various fields including econometrics, finance and actuaries [FV98].

In this paper, we introduce a BRDF representation based on copula functions. We focus on using a family of copulas called Archimedean copulas. The Archimedean representation has an attractive property in that it characterizes the multivariate copula through a single univariate function only.

It is shown that the proposed model meets the reciprocity property of reflection. Based on the emprical results, it is also shown that the Frank copula distribution, a member of the Archimedean family, provided the best approximation to measured BRDF data.

The next section gives a review of previous work. In Section 3, Archimedean copulas are presented, and in Section 4, approximating BRDF models by copula models are described. Section 5 describes procedures of model estimation. In Section 6, properties of the proposed model are discussed. Importance sampling is discussed in Section 7, and some empirical results are presented in Section 8. Finally, Section 9 is devoted for conclusions and future work.

2. Previous Work

BRDF models have received significant attention over the past decades in generating photo-realistic images. A wide range of models with varying degrees of complexity have been proposed to approximate surface reflectance. These models can be classified in three main groups: empirical models, physically-based models, and models based on linear combinations of some sets of basis functions [WLT04].

The most well known and the oldest empirical models developed to simulate the effects of the specular reflection are the Phong model [Pho75] and improved Phong model known as Blinn-Phong model [Bli77]. Ward [War92] introduced a simple formula to describe the isotropic and anisotropic reflectance distributions and fitted it to measured BRDF data. The Lafortune model [LFTG97] can be considered as a generalization of the Phong model, which can also represent a number of reflection properties such as non-Lambertian diffuse reflection, retro-reflection and Fresnel effect. Other empirical models include the models by Lewis [Lew94] and by Westlund and Meyer [WM01].

Physically based models mainly have been developed to simulate the reflectance properties for specular surfaces. The Torrance-Sparrow [TS67] and Cook-Torrance [CT81] models are the earliest models introduced in this category.

Assuming that the surface is composed of some microfacets, the Cook-Torrance model was designed to capture the reflectance properties of such surfaces. The Oren-Nayar model [ON94] was developed to approximate rough surfaces such as unglazed ceramics. A comprehensive but complicated model based on wave optics was introduced by

He-Torrance-Sillion-Greenberg [HTSG91]. Reflection models focusing on anisotropic materials also have been developed by Kajiya [Kaj85], Poulin and Fournier [PF90] and Ashikhmin et al. [APS00].

Approximating a continuous function by a linear combination of some set of basis functions is a widely used technique. Westin et al. [WAT92] have used spherical harmonics to represent BRDFs. As an alternative to employing spherical harmonics, Koendering et al. [KvDS96] used Zernike polynomials in a similar fashion to approximate BRDFs. Both the spherical harmonics and the Zernike polynomials require large numbers of basis functions to obtain accurate approximations, therefore the corresponding computational costs are high [Rus98]. Wavelets have also been used as basis functions for approximating BRDFs [SS95, KM99]. Matusik et al. [MPBM03] used principal component of the measured BRDF data as a basis function for BRDF representation. This class of linear representations provides good approximations to BRDF at the expense of involving large numbers of parameters. Using the response surface techniques, Ozturk et al. [OKBG08] presented an approach based on principal component transformations of some explanatory variables for approximating both isotropic and anisotropic reflectance for diffuse and glossy surfaces.

A natural way of modeling the surface reflection is to treat the BRDF in the context of probability theory. However, creating a 4D pdf to represent quickly-varying BRDFs accurately is difficult. Lawrence et al. [LRR04] decomposed the 4D function $f(\boldsymbol{\omega_i}, \boldsymbol{\omega_o})$ in Equation (3) into the sum of 2D functions such that one of the functions depends on $\boldsymbol{\omega_o}$ and the other function depends on the halfway direction. For a given outgoing direction vector $\boldsymbol{\omega_o}$ the second function further is factored into univariate pdfs. Based on this decomposition they have developed a general importance sampling algorithm. Similarly, using singular value decomposition separability of BRDF has been investigated by Fournier [Fou95] and Wang et al. [WTL04]. Using *homomorphic* factorization McCool et al. [MAA01] factorized BRDF into three 2D functions. Viewing the function $f(\boldsymbol{\omega_i}, \boldsymbol{\omega_o})$ as a pdf and assuming the outgoing direction vector $\boldsymbol{\omega_o}$ is known, Edwards et al. [EBJ*06] developed a new energy-conserving BRDF model which reflects many different reflectance effects.

3. Archimedean Copulas

In this section we give a brief description of Archimedean copulas. For an introduction to the theory of copulas we refer to [Nel06] and reviews to [WVS07].

A *copula* is defined to be a multivariate cumulative distribution function of the uniform random variables on the interval [0,1]. Suppose that the random variables $(X_1, X_2, \ldots, X_p)$ have a joint cumulative distribution function $F(x_1, x_2, \ldots, x_p)$ with marginal distribution functions $F_1(x_1), F_2(x_2), \ldots, F_p(x_p)$, respectively. It is well known that the random variables $U_i = F_i(x_i), i = 1, 2, \ldots, p$ are uniformly distributed on the interval [0,1]. No assumption is made about their independency. According to Sklar's theorem there exists a copula function C such that

$$\begin{aligned} C(F_1(x_1), F_2(x_2), \ldots, F_p(x_p)) &= C(u_1, u_2, \ldots, u_p) \\ &= F(x_1, x_2, \ldots, x_p), \end{aligned} \tag{4}$$

and, if the marginal distributions are continuous, then C is unique.

Copulas provide a general method for constructing multivariate distributions. As may be seen from the aforementioned equation, a copula separates the information on the marginal distributions and the information on the dependence from each other. A main advantage using a copula model is that the marginal distributions can be chosen independently after an appropriate copula model is selected to represent dependency between the variables.

The Archimedean family of copula distributions are defined by the distribution function

$$C(u_1, u_2, \ldots, u_p) = \psi^{-1}\{\psi(u_1) + \psi(u_2) + \cdots + \psi(u_p)\}, \tag{5}$$

where the function ψ called *generator*, is a convex, decreasing function on the interval [0,1], and the inverse function ψ^{-1} defined by the Laplace transform of a random variable X that is $\psi^{-1}(s) = E_X\{\exp(-sX)\}, s > 0$, where E_X is the expected value operator on the random variable X. For each choice of the generator, a different family of copulas is obtained. For example, the Frank family of distributions is generated from the following Laplace transform of the random variable X from the logarithmic series distribution:

$$\psi^{-1}(s) = \alpha^{-1} \ln[1 + \exp(s)(\exp(\alpha) - 1)], \alpha \neq 0, \tag{6}$$

for which the inverse solution is

$$\psi(t) = \ln \frac{\exp(\alpha t) - 1}{\exp(\alpha) - 1}. \tag{7}$$

A wide range of Archimedean family copula distributions has been developed. For a large selection of copula models we refer to [Joe97, Nel06]. Based on the Sklar theorem and Equation (4) it is clear that the copula distribution C and the marginal distributions $F_1, F_2, \ldots, F_p$ can be uniquely determined when the joint distribution function F is known. Conversely, a valid model can be constructed from a given parametric family of marginal distributions and a copula distribution [GF07]. A major advantage of this approach is that the marginal distributions are chosen independently and the dependence between the random variables is established by the copula function. For example, choosing the marginal distributions F_1 and F_2 as *normal* with parameters (μ, σ^2) and *exponential* with parameter λ, respectively, and the copula distribution from the Frank family, a bivariate distribution can be defined as

$$\begin{aligned} F(x_1, x_2) &= C(u_1, u_2) \\ &= \psi^{-1}\left\{\ln\frac{\exp(\alpha u_1)-1}{\exp(\alpha)-1} + \ln\frac{\exp(\alpha u_2)-1}{\exp(\alpha)-1}\right\} \\ &= \alpha^{-1}\ln\left[1+\frac{(\exp(\alpha u_1)-1)(\exp(\alpha u_2)-1)}{\exp(\alpha)-1}\right], \\ &\quad \alpha \neq 0, \end{aligned} \tag{8}$$

where $u_1 = F(x_1),\ u_2 = F(x_2)$ and α is the parameter of the copula distribution. The corresponding pdf can also be obtained by taking the second partial derivatives of F with respect to x_1 and x_2 as

$$\begin{aligned} p(x_1, x_2) &= \partial F(x_1, x_2)/\partial x_1 \partial x_2 \\ &= \frac{\alpha g_1(1+g_{u_1+u_2})}{(g_1 + g_{u_1} g_{u_2})^2}\prod_{j=1}^{2} f_j(x_j), \end{aligned} \tag{9}$$

where $g_t = e^{\alpha t} - 1$ and f_1, f_2 are the corresponding marginal *pdfs*.

4. Approximating the BRDFs by Archimedean Copulas

We now explain how the BRDF is approximated by a copula distribution. As was stated in the previous section, the function $\rho(\boldsymbol{\omega_i}, \boldsymbol{\omega_o})$ can be viewed as a multivariate pdf. In this work, we propose to approximate $\rho(\boldsymbol{\omega_i}, \boldsymbol{\omega_o})$ by choosing an appropriate Archimedean family of copula distributions.

Since we assume that the density function $\rho(\boldsymbol{\omega_i}, \boldsymbol{\omega_o})$ can be viewed as a multivariate pdf, then we can treat the measured BRDF values as the sampled densities after a simple transformation on the measured BRDFs such that the volume of the empirical density integrates to 1. For example, when the standard spherical coordinates are used to define the vectors $\boldsymbol{\omega_i}$ and $\boldsymbol{\omega_o}$ as explained in Section 1 and uniform spacing with unit intervals is used for the elevation and azimuth angles, namely θ and ϕ in the sampling process, the corresponding normalizing factor simply is the sum of all measured BRDFs. Hence the normalized BRDFs are considered to be the probability densities of a continuous distribution sampled at certain fixed points. Clearly, using this sample of densities an appropriate copula distribution from the Archimedean family can be fitted to approximate the corresponding "true" BRDF.

In this paper we consider that the BRDF is a 4D function that is

$$\rho(\boldsymbol{\omega_i}, \boldsymbol{\omega_o}) = \rho(\theta_i, \phi_i, \theta_o, \phi_o). \tag{10}$$

In a typical application the outgoing direction $\boldsymbol{\omega_o} = (\theta_o, \phi_o)$ is assumed to be known. In this case the BRDF becomes a bivariate function of only θ_i and ϕ_i. However, a number of representations including the ones based on the half-angle vector

$$\mathbf{h} = \frac{\boldsymbol{\omega_i} + \boldsymbol{\omega_o}}{\|\boldsymbol{\omega_i} + \boldsymbol{\omega_o}\|}, \tag{11}$$

have been reported to yield more *visually plausible* results than BRDFs based on the incident vector defined in standard spherical coordinates [Rus98, APS00, SAS05, EBJ*06]. For example, if a representation proposed by Rusinkiewicz [Rus98] is chosen instead of the usual parameterization in terms of angles of incidence and reflection, then the *halfway* vector $\mathbf{h} = (\theta_h, \phi_h)$ and the *difference* vector $\mathbf{d} = (\theta_d, \phi_d)$ can be used in the pdf function in Equation (10) instead of the spherical coordinates for $\boldsymbol{\omega_i}$ and $\boldsymbol{\omega_o}$. It is noted that isotropic BRDFs are independent of ϕ_h in this coordinate system and the corresponding joint distribution can be represented by a three-variate pdf. The proposed approach based on 4D copula distributions also can be used for modeling anisotropic BRDFs in a straightforward way. Generally, dimension of a copula is determined depending on the nature of the problem studied. For example, if the wavelength is also considered for describing BRDF then an additional dimension to account for this new factor should be considered in the corresponding copula model.

Let $F_1(\theta_h)$, $F_2(\theta_d)$ and $F_3(\phi_d)$ denote the marginal cumulative distribution functions based on the representation proposed by Rusinkiewicz [Rus98] for an isotropic BRDF. Emprical marginal cumulative distributions of the intensites for a number of isotropic BRDFs are illustrated in Figure 1. Let also $f_1(\theta_h)$, $f_2(\theta_d)$ and $f_3(\phi_d)$ denote the corresponding marginal density functions. A full BRDF model that we

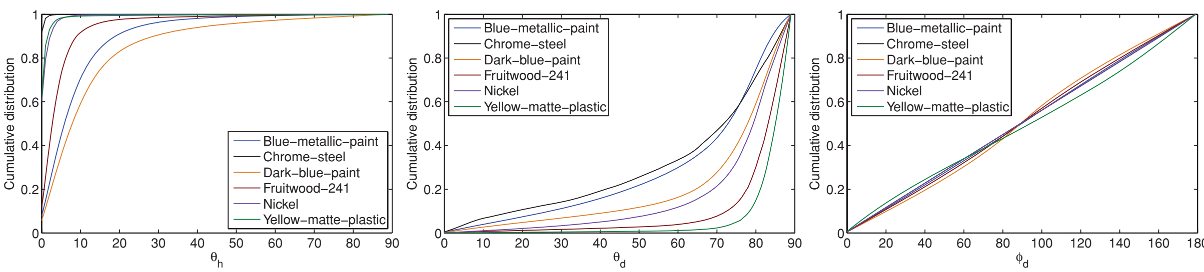


Figure 1: *Emprical marginal cumulative distribution of θ_h, θ_d and ϕ_d for various materials.*

employ in this paper can be expressed as

$$\rho(\theta_h, \theta_d, \phi_d) = Kc(u_1, u_2, u_3) f_1(\theta_h) f_2(\theta_d) f_3(\phi_d), \quad (12)$$

where $u_1 = F_1(\theta_h),\ u_2 = F_2(\theta_d),\ u_3 = F_3(\phi_d),\ c$ is a three-dimensional (3D) copula pdf and K is the scaling parameter. The parameter(s) of the copula distribution and marginal density functions usually are unknown and should be estimated from the measured BRDF data. When the Frank copula is used to represent an isotropic BRDF the corresponding pdf can be easily obtained by extending bivariate pdf to a 3D pdf in Equation (9) as

$$p(\mathbf{x}) = \alpha^2 g_1^2 (1 + g_{u_1+u_2+u_3}) \frac{g_1^2 - g_{u_1} g_{u_2} g_{u_3}}{\left(g_1^2 + g_{u_1} g_{u_2} g_{u_3}\right)^3} \prod_{j=1}^{3} f_j(x_j), \quad (13)$$

where $\mathbf{x} = (x_1, x_2, x_3) = (\theta_h, \theta_d, \phi_d)$, $g_t = e^{\alpha t} - 1$ and $f_1,\ f_2,\ f_3$ are the marginal pdfs of $\theta_h,\ \theta_d,\ \phi_d$, respectively.

Note that the Frank copula has only one unknown parameter.

5. Estimation

In this section, we explain the estimation procedure of a copula model for isotropic BRDF using the half-angle parameterization of Rusinkiewicz. Extension of the proposed approach for other parameterizations and for anisotropic case can be made in a similar way.

We first define our sample which is used for estimating the proposed copula model. Suppose that a measured BRDF data is obtained at a dense grid with every $\delta_{\theta_h},\ \delta_{\theta_d}$ and δ_{ϕ_d} degrees of spacing in the intervals $0 \le \theta_h < 90,\ 0 \le \theta_d < 90$ and $0 \le \phi_d < 180$, respectively. The number of samples taken along $\theta_h,\ \theta_d$ and ϕ_d directions are $n = 90/\delta_{\theta_h},\ m = 90/\delta_{\theta_d}$ and $r = 180/\delta_{\phi_d}$, respectively, giving a total number of data points of $90 \times 90 \times 180/(\delta_{\theta_h}\delta_{\theta_d}\delta_{\phi_d})$. It is easy to see that, when a uniform spacing with unit length of 1° in all directions is used, then the total number of bins will be $90 \times 90 \times 180 = 1,458,000$ [MPBM03].

Next we normalize the BRDFs measured at each bin as

$$b_{ijk} = \frac{b^*_{ijk}}{K}, \quad (14)$$

where b^*_{ijk} is the measured BRDF at a bin whose coordinates are $(i, j, k), i = 1, 2, \ldots, n; j = 1, 2, \ldots, m; k = 1, 2, \ldots, r$ and $K = \delta_{\theta_h}\delta_{\theta_d}\delta_{\phi_d} \sum_{i=1}^{n} \sum_{j=1}^{m} \sum_{k=1}^{r} b^*_{ijk}$ is the scaling factor. The data matrix for the normalized BRDFs is organized as follows

$$\mathbf{D} = \left\{ b_{ijk}, \theta_h^{(i)}, \theta_d^{(j)}, \phi_d^{(k)} \right\}, \quad (15)$$

where the superscripts in the brackets stand for the bin number of the corresponding angle.

We express the copula pdf c given in Equation (12) with a single parameter α for the aforementioned data set as

$$c(u_1, u_2, u_3; \alpha) = c(F_1(\theta_h), F_2(\theta_d), F_3(\phi_d); \alpha), \quad (16)$$

where $F_1,\ F_2$ and F_3 are the cumulative marginal distribution functions of $\theta_h,\ \theta_d$ and ϕ_d, respectively. In this model the parameter α of the given copula pdf and the marginal distributions are unknown and have to be estimated from the sample.

Various strategies have been proposed for parameter estimations of copula distributions. Rank-based nonparametric estimators, including the inversions of Kendall's *tau* and Spearman's *rho*, are the well known estimators although there has been no complete consensus in the statistical community about them [GF07]. A popular approach referred to as the *inference from margins* (IFS) proposed by Joe [Joe97] employs the two-step maximum likelihood (ML) estimation procedure. ML estimates of the parameters of the marginal distributions are obtained from the sample first and then the log-likelihood function is maximized with respect to the dependence parameter α.

However, neither of these methods can be used directly when the random sample consists of the BRDF data because it differs from a standard random sample. As explained aforementioned, BRDF measurements are obtained at fixed points and they are scaled so that the resulting sample represents the density function values evaluated at these fixed points. Therefore the problem of estimating a three-dimensional probability distribution is reduced to a standard least squares estimation problem.

We propose to use a modification of the IFS method for parameter estimation of the copula distribution for this special case. First, we obtain empirical estimates of the marginal distributions from the BRDF sample and retrieve the required function value whenever needed. Linear interpolation techniques are used to determine the quantiles of the empirical distributions. Then the estimate of α is obtained through the minimization of the objective function of the form

$$S(\alpha) = \sum_{i=1}^{n}\sum_{j=1}^{m}\sum_{k=1}^{r} \left\{ b_{ijk} - c\left(F_1^{(i)}, F_2^{(j)}, F_3^{(k)}; \alpha\right) f_1^{(i)} f_2^{(j)} f_3^{(k)} \right\}^2 \quad (17)$$

where $f_1^{(i)} = \sum_{j=1}^{m}\sum_{k=1}^{r} b_{ijk},\ f_2^{(j)} = \sum_{i=1}^{n}\sum_{k=1}^{r} b_{ijk}$, $f_3^{(k)} = \sum_{i=1}^{n}\sum_{j=1}^{m} b_{ijk}$ and $F_1,\ F_2,\ F_3$ are the corresponding estimates of the cumulative marginal distribution functions of $\theta_h,\ \theta_d$ and ϕ_d, respectively.

It is noted that after the marginal distributions are estimated, a standard non-linear least squares estimation technique can be used to obtain the estimate of α.

A pseudo-code for our proposed estimation procedure based on Rusinkiewicz parameterization for isotropic BRDFs is presented in the Appendix (Algorithm 1, Algorithm 2 and Algorithm 3).

6. Properties of the Model

The reciprocity and the conservation of energy are two important properties of an ideal BRDF. Generally, it is difficult to construct a general purpose BRDF model based on a closed form of a mathematical expression that satisfies both the reciprocity and energy conservation properties. However, most BRDF models that have been used to simulate reflectance of a material violate at least reciprocity or energy conservation [EBJ*06]. Our copula model satisfies only the reciprocity property of BRDF for the two representations, namely the standard spherical coordinate system and the Rusinkiewicz system. Reciprocity is expressed by the equation

$$\rho(\boldsymbol{\omega_i}, \boldsymbol{\omega_o}) = \rho(\boldsymbol{\omega_o}, \boldsymbol{\omega_i}). \tag{18}$$

Formally, an Archimedean copula distribution is defined by Equation (5). It is clear from this expression that the distribution function $\psi^{-1}(\cdot)$ is independent of the ordering of its arguments, that is, the function value is not changed when the direction vectors $\boldsymbol{\omega_o}$ and $\boldsymbol{\omega_i}$ defined in standard spherical coordinates is interchanged. If the variables based on half-angle parameterization of Rusinkiewicz is used, then interchanging $\boldsymbol{\omega_o}$ and $\boldsymbol{\omega_i}$ will not satisfy the reciprocity property unless the underlying system is enforced by the following simple translation for ϕ_d [Rus98]

$$\phi_d = \phi_d + \pi. \tag{19}$$

Therefore the proposed model can be considered to be reciprocal provided that the above transformation is used.

7. Importance Sampling

Importance sampling is a variance reduction technique and it is well studied (see, e.g. [JAD*03]).

For our representation, deriving an exact distribution is not straightforward. Based on Equation (1), the incident illumination can be expressed in terms of 4D original BRDF by

$$L_o(\theta_o, \phi_o) = \int_0^{2\pi} \int_0^{\pi/2} L_i(\theta_i, \phi_i)\rho(\theta_i, \phi_i, \theta_o, \phi_o) \cos\theta_i d\omega_i, \tag{20}$$

where $d\omega_i = \sin\theta_i d\theta_i d\phi_i$ [War92]. Let T denote a transformation between the original BRDF and the new representation denoted by $(\theta_h,\ \phi_h,\ \theta_d,\ \phi_d)$. The distribution can be expressed in terms of the new representation by

$$h(\theta_i, \phi_i, \theta_o, \phi_o) = g(\theta_h, \phi_h, \theta_d, \phi_d)|J|, \tag{21}$$

where J is the Jacobian of the underlying change of variables. Conditional distribution of θ_i and ϕ_i given that θ_o and ϕ_o is given by

$$f(\theta_i, \phi_i|\theta_o, \phi_o) = \frac{g(\theta_h, \phi_h, \theta_d, \phi_d)|J|}{p(\theta_o, \phi_o)}, \tag{22}$$

where p is the bivariate marginal pdf of θ_o and ϕ_o. Based on this definition, the Monte Carlo estimator for the incident illumination integral can be written as

$$\frac{1}{n}\sum L_i(\theta_i, \phi_i)\frac{\rho(\theta_i, \phi_i, \theta_o, \phi_o)\cos\theta_i \sin\theta_i}{f(\theta_i, \phi_i|\theta_o, \phi_o)}. \tag{23}$$

In order to develop a sampling procedure, we need to generate incident vector $\boldsymbol{\omega_i} = (\theta_i, \phi_i)$ for a given outgoing direction vector $\boldsymbol{\omega_o} = (\theta_o, \phi_o)$, from the distribution f. Standard random variate generation techniques such as the rejection method can be employed to generate incident vectors. 2D and 4D Frank copula can be used for approximating the distributions f and g, respectively.

The computational cost of the above random vector generation procedure is very expensive since generating random variates from the distribution f is not easy. However, some approximations can be made to simplify the corresponding functions and reduce the time required for the Monte Carlo integration.

8. Results

To investigate some empirical properties of the proposed model, we have used a data set based on 30 isotropic materials acquired by Matusik et al. [MPBM03] from the MERL MIT database. These samples were chosen to represent a wide range of materials with different diffuse and specular reflection properties. Some materials displaying extreme reflection properties such as chrome-steel, which is known to be the most specular, and fruitwood-241, which present some non-standard behavior, also were included in the data set. Using half-angle parameterization of Rusinkiewicz, the BRDF measurements were provided in 3D angular space defined by the vector $(\theta_h, \theta_d, \phi_d)$. 1,458,000 BRDF measurements based on uniform spacing with unit length of 1° in all directions were obtained for each sample material and for each color channel.

Examination of sample data showed that the marginal distribution of θ_h is extremely skewed, especially for specular materials. This situation is demonstrated for chrome-steel, nickel and yellow-matte-plastic in Figure 1. Our empirical results have shown that copula distributions do not provide satisfactory approximations when marginal distributions are extremely skewed. This situation was observed mostly for specular BRDFs. For these kinds of extremely skewed cases an optimal subdivision of the BRDF sample could be made to improve the accuracy of the copula approximation. It is noted in Figure 1 that the distribution of ϕ_d is approximately uniform. Therefore, the conditional joint distribution of θ_h and ϕ_d given that θ_d will be dominated by the marginal

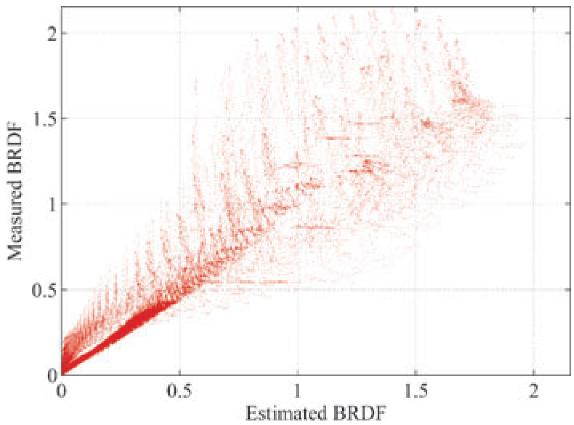


Figure 2: *Plots of measured BRDFs of blue-metallic-paint against the estimated values based on Frank copula distribution.*

distribution of θ_h. Dividing the BRDF sample into slices along the angle θ_d and fitting a copula distribution to each of these slices (sub-samples) should reduce the approximation error at the expense of increasing the computational cost. For practical considerations we divided the BRDF sample into sub-samples along the angle θ_d with uniform intervals. The number of divisions depends on the material used. For this special case, we used 6 divisions and each sub-sample corresponds to a sub-interval of θ_d with length $90^\circ/6 = 15^\circ$. Finally we fitted copula distribution for each sub-sample.

In order to give insight into how well the proposed model represents BRDF, a 3D Frank copula model was fitted to measured data of *blue-metallic-paint*. Figure 2 shows resulting plots of measured BRDFs against the estimated BRDFs for red channel. It is seen from the figure that there is a good agreement between the estimated and measured BRDF values for this color channel. The existence of some outliers can be seen clearly. Ngan et al. [NDM05] also noted this situation and they ignored the data with incident or outgoing angle larger than 80 degrees in their analysis.

Identifying the best candidate family of copula distributions is an important issue in this application. However, we eliminated some of the families at the beginning because of some theoretical limitations. For example, for some of the distributions, a global minimum could not be reached in the acceptable range of the corresponding parameter during the optimization process.

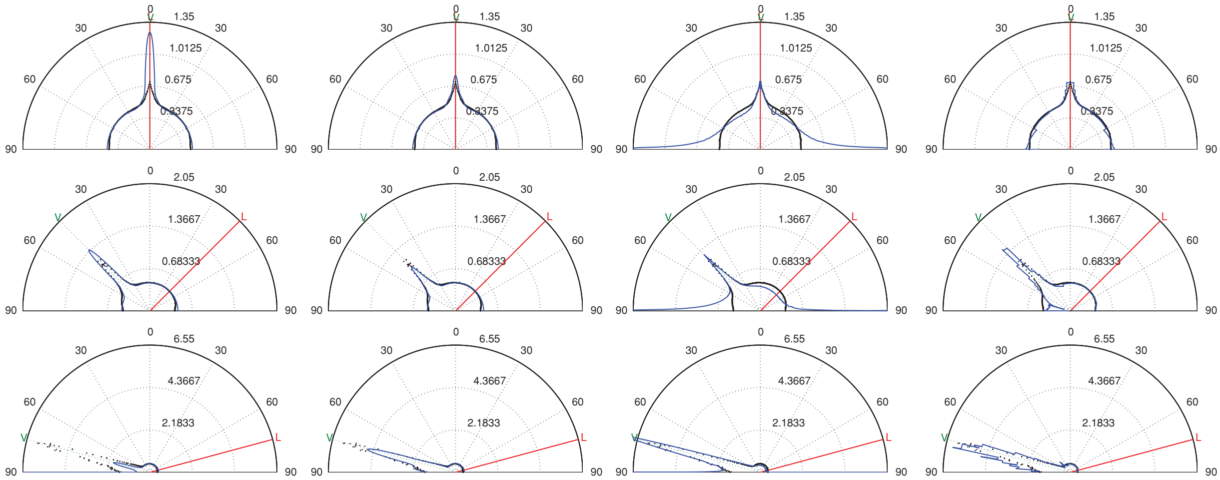
The remaining Archimedean models, including the Clayton, Frank, Gumbel-Hougaard families of copulas [Nel06] and BB1, BB2, BB3, BB6 and BB7 class of distributions [Joe97], were considered. The IFS method as explained in Section 5 was used for estimating the dependence parameter. A constrained non-linear optimization technique is applied to minimize the objective function in Equation (17) using FMINCON [WMNO06] in MATLAB library. FMINCON is a constrained nonlinear optimization algorithm that attempts to find a minimum of a function of several variables starting at an initial estimate. This algorithm does not guarantee a convergence toward a global minimum. However, our objective function in Equation (17) has one variable only, and optimum solutions have been obtained without facing any difficulty. To identify the best fitting model, agreements between the fitted model and the measured BRDFs were examined. It was interesting to note that the Frank distribution was identified as the best model for all materials considered and for all color channels.

Another graphical comparison based on the polar plots of various models, including the Ward, the Cook-Torrance, the Lawrence et al. and Frank copula models, were made. Fitted and measured BRDFs are plotted in the incidence plane for $\theta = 0, 45, 75$ degrees. Figure 3 illustrates the underlying polar plots based on *yellow-matte-plastic*. The relative quality of the fitted models can be compared for this material. It is noted that the Frank copula model performed well for these cases.

Renderings of spheres based on the BRDF measurements were generated using the Frank copula are presented in Figure 4. Direct illumination was used for rendering. Spheres in the top row and in the middle row are based on the measured BRDF data and the Frank copula models, respectively. The differences between the first row images and the second row images are shown in the bottom row. All difference images are standardized using the same scaling factor to provide a better visual comparison. It is seen from this figure that the Frank copula model provided extremely good matching with the real images in all cases except chrome-steel. Chrome-steel is known to be a highly specular material and many BRDF models have failed to provide satisfactory representations [NDM05].

To assess the fitting quality of the model quantitatively we calculated the Peak Signal-to-Noise Ratio (PSNR) values for each color channel and obtained their averages [LC07]. For comparision, we have fitted several BRDF models using Ngan et al.'s [NDM05] fitting procedures. The PSNR values for different models and materials are presented in Table 1 and Figure 5. The results in the table and figure indicate that for these materials, the Frank copula model has yielded the highest PSNR values in most cases. The Frank copula model did not perform well only for highly specular material, namely red-specular-plastic and for diffuse materials, namely beige-fabric and white-diffuse-bball. The significant PSNR differences between the proposed Frank copula model and the others explain the higher quality of the underlying model.

To make a further comparison, we obtained renderings of a scene based on three different BRDF models, namely the Ward model [War92], the Lawrence et al. representation [LRR04] and our model (with Frank distribution). Measured BRDF data from Matusik et al. [MPBM03] was used and the



(a) Ward (b) Cook-Torrance (c) Lawrence et al. (d) Frank copula

Figure 3: *Polar plots of various fitted models against the measured BRDF (black dashed lines) of yellow-matte-plastic in the incidence plane. Columns from left to right: (a) the Ward, (b) the Cook-Torrance, (c) the Lawrence et al. and (d) the Frank copula models. Rows from top to bottom: $\theta = 0, 45, 75$ degrees. Cubic root applied for visualization purpose.*

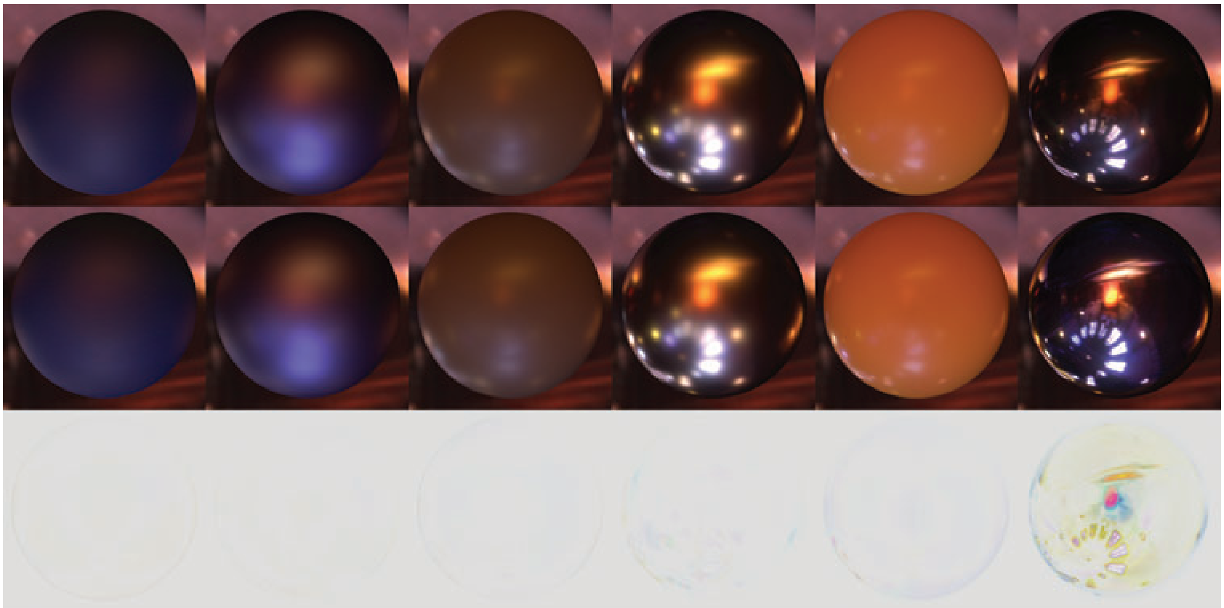


Figure 4: *Various spheres rendered with our Frank copula model using different materials. Columns left to right: dark-blue-paint, blue-metallic-paint, fruitwood-241, nickel, yellow-matte-plastic and chrome-steel. Rows top to bottom: Reference images rendered using measured data; images rendered using Frank copula model based on $(\theta_h,\ \theta_d,\ \phi_d)$ and difference images.*

resulting images, shown in Figure 6, obtained in a similar way as given by Edwards et al. [EBJ*06] who refer to a similar figure from Lawrence et al. [LRR04]. The globally illuminated scene rendered at 4096 samples per pixel. Insets in the figure represent the difference images between the real image (top left image in Figure 6) and the corresponding rendered image. By visual comparisons of the images based on considering both the insets and the images themselves in this figure, it is seen that the Frank copula model has provided the best representation. The corresponding PSNR values for the Ward model, for the Lawrence et al. representation and for our model were found to be 25.77, 31.55 and 38.53,

Table 1: *PSNR values for the three BRDF models based on Figure 4.*

BRDF Model	Blue-metallic-paint	Nickel	Yellow-matte-plastic
Frank copula	46.39	43.16	42.12
Lawrence et al.	37.85	32.94	29.71
Ward	32.85	28.52	33.54

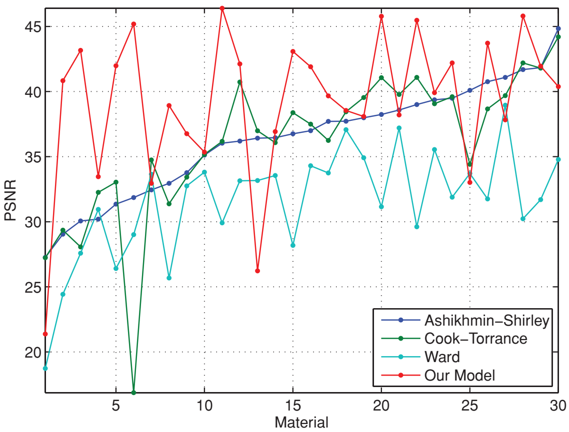


Figure 5: *The PSNR values of the Ashikhmin-Shirley, the Cook-Torrance, the Ward and our Frank copula models. The BRDFs are sorted in the PSNRs of the Ashikhmin-Shirley model (Blue) for visualization purpose.*

respectively. These results also give some idea about the representational ability of our Frank copula model.

The Archimedean copula distributions considered in this paper are characterized through a single nonlinear parameter and the marginal distributions. If the marginal distributions were known, we would need to store a single parameter value only. Another possibility is to identify the corresponding marginal distributions and estimate their parameters from the sample data. In this paper we stored the original data for each empirical marginal distribution. This is, of course, the worst case of our approach in terms of storage needs. We compare the storage need of the Frank copula model with one of its competitors, namely the Lawrence et al. representation. The results obtained for each of the three different materials are presented in Table 2. The complete Frank copula representation requires less data storage than its competitors in all cases.

A similar comparison was performed on rendering times of the models based on Figure 4, which has 1024 samples per pixel. The results are presented in Table 3. Rendering times were acquired on a Pentium Core2Duo 2.33 GHz computer with 3 GB memory. As seen from Table 3, the rendering times of the Frank copula model is higher than rendering times of the other models.

Figure 6: *Top left: Reference image using measured BRDF data, Top right: The Lawrence et al.'s representation, Bottom left: The Ward model based on two specular lobes, Bottom right: The Frank copula model.*

Table 2: *Required storage spaces by the three BRDF representations for various materials. Rendering data are prepared in binary double precision for all BRDF representations.*

BRDF Model	Blue-metallic-paint	Nickel	Yellow-matte-plastic
Measured	33.4MB	33.4MB	33.4MB
Lawrence et al.	139.0KB	96.5KB	331.9KB
Frank copula	40.4KB	40.4KB	40.4KB

Table 3: *Rendering times (in seconds) of BRDF models.*

BRDF Model	Blue-metallic-paint	Nickel	Yellow-matte-plastic
Measured	2979.6	2689.3	2958.1
Frank copula	3167.0	3154.9	3172.1
Lawrence et al.	2960.0	2840.4	2751.6
Ward	2041.3	2198.2	2219.8

Finally we demonstrate the reciprocity property of the proposed model in Figure 7. Images presented in Figure 7 were obtained by interchanging the incident and outgoing

Figure 7: *Left: scene rendered using Frank copula model. Middle: scene rendered using the same Frank copula model with incoming and outgoing vectors exchanged. Right: A difference between left and middle images.*

directions and the renderings of the objects for each case are shown in the figure.

9. Conclusions and Future Work

In this paper, we have introduced a novel approach for modeling surface reflection. The proposed model can also be referred to as a compression technique for the sampled BRDF data. This model essentially is based on treating BRDF as a multivariate probability distribution and measured BRDF data as a sample from this distribution. The Frank distribution from the Archimedean family of copula distributions has been employed for this purpose. Using a well-known data set, empirically we have shown that Frank copula provided extremely well approximations in most cases.

Empirical marginal distributions were stored and used for nonparametric estimation of the underlying distributions (in this sense the proposed technique for BRDF representation can also be considered to be a compression procedure of BRDF data).

We have outlined an importance sampling technique that can be used for the evaluation of the lighting model. However, implementation of the proposed technique is not easy and a number of numerical problems need to be solved. Our research is being continued along this line.

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Appendix

Algorithm 1 prepares data for model fitting. It receives sampled measured BRDF data as a parameter and returns normalized measured BRDF data, F_1, F_2, F_3 and normalization factors. F_1, F_2, F_3 and normalization factors are also used in rendering.

Algorithm 2 is the fitting objective function. This algorithm also applies to the Algorithm 3, which is the 3D Frank copula pdf. Algorithm 4 is based on our rendering algorithm for 3D Frank copula model based on $(\theta_h, \theta_d, \phi_d)$. Algorithm 4 uses F_1, F_2, F_3, normalization factors and estimated αs.

These algorithms consider only one color channel. They should be used for each channel separately.

Algorithm 1: prepareFittingData(measBRDF)

let partθ_d be the number of partitions
for $i = 0$ **to** partθ_d
    let normBRDF[i] be the normalized measured brdf data such that
    calculate nCoeff[i] = **sum**(measBRDF[i]) and normBRDF[i] = measBRDF[i] / nCoeff[i]
    calculate $f_1[i](\theta_h)$ from normBRDF[i]
    calculate $f_2[i](\theta_d)$ from normBRDF[i]
    calculate $f_3[i](\phi_d)$ from normBRDF[i]
    /* $f_1(\theta_h),\ f_2(\theta_d),\ f_3(\phi_d)$ are 1D pdfs*/
    calculate $F_1[i](\theta_h) =$ **cumsum**$(f_1[i](\theta_h))$
    calculate $F_2[i](\theta_d) =$ **cumsum**$(f_2[i](\theta_d))$
    calculate $F_3[i](\phi_d) =$ **cumsum**$(f_3[i](\phi_d))$
    /* $F_1(\theta_h),\ F_2(\theta_d),\ F_3(\phi_d)$ are 1D cdfs*/
end for
return normBRDF, $F_1,\ F_2,\ F_3$ and nCoeff

Algorithm 2: fittingObjective($\alpha,\ F_1,\ F_2,\ F_3$, normBRDF, partNumber)

set $i =$ partNumber
set error = 0
for $j = 0$ **to** sampleSizeN
    find $u_1 = F_1[i](\theta_h[j])$
    find $u_2 = F_2[i](\theta_d[j])$
    find $u_3 = F_3[i](\phi_d[j])$
    find $c =$ frank3DPdf$(u_1,\ u_2,\ u_3,\ \alpha)$
    find $f_1 = F_1'[i](\theta_h[j])$
    find $f_2 = F_2'[i](\theta_d[j])$
    find $f_3 = F_3'[i](\phi_d[j])$
    calculate estBRDF $= c \times f_1 \times f_2 \times f_3$
    calculate error = error + **pow**((estBRDF− normBRDF[i][j]), 2)
end for
return error

Algorithm 3: frank3DPdf($u_1,\ u_2,\ u_3,\ \alpha$)

calculate $g_1^2 = \textbf{pow}((\textbf{exp}(\alpha)-1),2)$
calculate $g_u = \textbf{exp}(\alpha \times u_1)-1$
calculate $g_v = \textbf{exp}(\alpha \times u_2)-1$
calculate $g_y = \textbf{exp}(\alpha \times u_3)-1$
calculate $g_{u+v+y} = \textbf{exp}(\alpha\times(u_1+u_2+u_3))-1$
calculate pdf $= \textbf{pow}(\alpha, 2)\times(1+g_{u+v+y})\times g_1^2\times$
$g_1^2 - g_u \times g_v \times g_y)/(\textbf{pow}((g_1^2 + g_u \times g_v \times g_y), 3))$
return pdf

Algorithm 4: renderBrdfCopula($\boldsymbol{\omega_i},\ \boldsymbol{\omega_o}$)

get $(\theta_i,\ \phi_i)$ angles from $\boldsymbol{\omega_i}$
get $(\theta_o,\ \phi_o)$ angles from $\boldsymbol{\omega_o}$
apply Rusinkiewicz coordinate transformation and get
$(\theta_h,\ \phi_d,\ \theta_d,\ \phi_d)$ angles from $(\theta_i,\ \phi_i,\ \theta_o,\ \phi_o)$ angles
/*enforce reciprocity: $\phi_d \leftarrow \phi_d + \pi$*/
if($\phi_d < 0$) **then**
    $\phi_d = \phi_d + \pi$
end if
let partθ_d be the number of partitions
find proper partition index $i = (\theta_d/(\pi/2/\text{part}\theta_d))$
recalculate $\theta_d = \theta_d - i \times (\pi/2/\text{part}\theta_d)$
find $u_1 = F_1[i](\theta_h)$
find $u_2 = F_2[i](\theta_d)$
find $u_3 = F_3[i](\phi_d)$
/*$F_1(\theta_h),\ F_2(\theta_d),\ F_3(\phi_d)$ are 1D cdfs*/
find c = frank3DPdf($u_1,\ u_2,\ u_3,\ \alpha[i]$)
find $f_1 = F_1'[i](\theta_h)$
find $f_2 = F_2'[i](\theta_d)$
find $f_3 = F_3'[i](\phi_d)$
/*$F_1'(\theta_h),\ F_2'(\theta_d),\ F_3'(\phi_d)$ are 1D pdfs*/
let nCoeff[i] be sum of the measured brdfs
calculate brdf $= c \times f_1 \times f_2 \times f_3 \times$nCoeff[$i$]
return brdf

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